

# GAUSSIAN INTEGRAL MEANS OF ENTIRE FUNCTIONS

CHUNJIE WANG AND JIE XIAO

**ABSTRACT.** For an entire function  $f : \mathbb{C} \mapsto \mathbb{C}$  and a triple  $(p, \alpha, r) \in (0, \infty) \times (-\infty, \infty) \times (0, \infty]$ , the Gaussian integral means of  $f$  (with respect to the area measure  $dA$ ) is defined by

$$M_{p,\alpha}(f, r) = \left( \int_{|z|<r} e^{-\alpha|z|^2} dA(z) \right)^{-1} \int_{|z|<r} |f(z)|^p e^{-\alpha|z|^2} dA(z).$$

Via deriving a maximum principle for  $M_{p,\alpha}(f, r)$ , we establish not only Fock-Sobolev trace inequalities associated with  $M_{p,p/2}(z^m f(z), \infty)$  (as  $m = 0, 1, 2, \dots$ ), but also convexities of  $r \mapsto \ln M_{p,\alpha}(z^m, r)$  and  $r \mapsto M_{2,\alpha<0}(f, r)$  in  $\ln r$  with  $0 < r < \infty$ .

## 1. INTRODUCTION

Let  $dA$  be the Euclidean area measure on the finite complex plane  $\mathbb{C}$ . Suppose  $\alpha$  is real and  $0 < p < \infty$ . For any entire function  $f : \mathbb{C} \mapsto \mathbb{C}$ , we consider its Gaussian integral means

$$M_{p,\alpha}(f, r) = \frac{\int_{|z|<r} |f(z)|^p e^{-\alpha|z|^2} dA(z)}{\int_{|z|<r} e^{-\alpha|z|^2} dA(z)} \quad \forall \quad r \in (0, \infty).$$

Upon writing

$$\begin{cases} M(r) = \int_0^{2\pi} |f(re^{i\theta})|^p d\theta; \\ v(r) = re^{-\alpha r^2}; \\ i = \sqrt{-1} - \text{the imaginary unit,} \end{cases}$$

we get

$$\frac{d}{dr} M_{p,\alpha}(f, r) = \frac{v(r) \int_0^r (M(r) - M(s)) v(s) ds}{2\pi \left( \int_0^r v(s) ds \right)^2} \geq 0,$$

and hence the function  $r \mapsto M_{p,\alpha}(f, r)$  is strictly increasing on  $(0, \infty)$  unless  $f$  is constant. Consequently, letting  $r \rightarrow 0$  and  $r \rightarrow \infty$  in  $M_{p,\alpha}(f, r)$

---

2000 *Mathematics Subject Classification.* Primary 30C80, 30H20, 52A38, 53C43.

*Key words and phrases.* Maximum principle, trace inequality, logarithmic convexity, Fock-Sobolev space.

This work was in part supported by NSERC of Canada and completed during the first-named author's visit (2012.9-12) to Memorial University.

respectively, we find the following maximum principle for  $r \in (0, \infty)$ :

$$\begin{aligned} |f(0)|^p &= M_{p,\alpha}(f, 0) \leq M_{p,\alpha}(f, r) \\ &\leq M_{p,\alpha}(f, \infty) = \frac{\int_{\mathbb{C}} |f(z)|^p e^{-\alpha|z|^2} dA(z)}{\int_{\mathbb{C}} e^{-\alpha|z|^2} dA(z)} \end{aligned}$$

with equality if and only if  $f$  is a constant.

Besides the above maximum principle we are here motivated mainly by [15, 6, 7, 13, 12, 14, 2] to take a further look at the Gaussian integral means  $M_{p,\alpha}(f, r)$  from two perspectives. The first is to treat the last inequality as a space embedding: if  $d\mu_r(z) = 1_{|z|<r} dA(z)$  (with  $1_E$  being the characteristic function of  $E \subset \mathbb{C}$ ) then

$$\int_{\mathbb{C}} |f(z) e^{-\frac{|z|^2}{2}}|^p d\mu_r(z) \leq \left( \int_{|z|<r} e^{-\frac{p|z|^2}{2}} dA(z) \right) M_{p,p/2}(f, \infty).$$

Such an interpretation leads to characterizing a given nonnegative Borel measure  $\mu$  on  $\mathbb{C}$  such that the following Fock-Sobolev trace inequality

$$\begin{aligned} \|f\|_{L^q(\mathbb{C}, \mu)} &\equiv \left( \int_{\mathbb{C}} |f(z) e^{-\frac{|z|^2}{2}}|^q d\mu(z) \right)^{\frac{1}{q}} \\ &\lesssim \left( M_{p,p/2}(z^m f(z), \infty) \right)^{\frac{1}{p}} \\ &\approx \left( \int_{\mathbb{C}} |z^m f(z) e^{-\frac{|z|^2}{2}}|^p dA(z) \right)^{\frac{1}{p}} \\ &\equiv \|f\|_{\mathcal{F}^{p,m}} \end{aligned}$$

holds for all holomorphic functions  $f : \mathbb{C} \mapsto \mathbb{C}$  in  $\mathcal{F}^{p,m}$ . In the above and below:

- $0 < p, q < \infty$ ;
- $X \lesssim Y$  (i.e.  $Y \gtrsim X$ ) means that there is a constant  $c > 0$  such that  $X \leq cY$  - moreover -  $X \approx Y$  is equivalent to  $X \lesssim Y \lesssim X$ ;
- $m$  is nonnegative integer;
- $\mathcal{F}^p = \mathcal{F}^{p,0}$  and  $\mathcal{F}^{p,m}$  stand for the so-called Fock space and Fock-Sobolev space of order  $m \geq 1$  respectively. Interestingly, for an entire function  $f : \mathbb{C} \mapsto \mathbb{C}$  one has (cf. [2]):

$$f \in \mathcal{F}^{p,m} \iff |f(0)| + \dots + |f^{(m-1)}(0)| + \|f^{(m)}\|_{\mathcal{F}^{p,0}} < \infty.$$

- $B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$  is the Euclidean disk centered at  $a \in \mathbb{C}$  with radius  $r > 0$ .

As stated in Theorem 3 of Section 2, the above-required measure is fully determined by

$$\begin{cases} \sup_{a \in \mathbb{C}} \frac{\mu(B(a,r))}{(1+|a|)^{qm}} < \infty & \text{as } 0 < p \leq q < \infty; \\ \int_{\mathbb{C}} \left( \frac{\mu(B(a,r))}{(1+|a|)^{qm}} \right)^{\frac{p}{p-q}} dA(a) < \infty & \text{as } 0 < q < p < \infty. \end{cases}$$

As a particularly interesting and natural by-product of this characterization, we can also use the Taylor expansion of an entire function at the origin to get the optimal Gaussian Poincaré inequality (see [8, (1.6)] as well as [5, p. 115] and [16, Theorem 1] for the endpoint case corresponding to  $f \in \mathcal{F}^{1,1}$  with  $f(0) = 0$ )

$$\int_{\mathbb{C}} |f(z)e^{-\frac{|z|^2}{2}}|^2 dA(z) - \pi|f(0)|^2 \leq \int_{\mathbb{C}} |f'(z)e^{-\frac{|z|^2}{2}}|^2 dA(z) \quad \forall f \in \mathcal{F}^{2,1}$$

which, plus the foregoing maximum-principle-based estimate (cf. [2, (1)])

$$|f'(z)|e^{-\frac{|z|^2}{2}} \leq (2\pi)^{-1} \int_{\mathbb{C}} |f'(z)e^{-\frac{|z|^2}{2}}| dA(z) \quad \forall f \in \mathcal{F}^{1,1},$$

derives the following Gaussian isoperimetric-Sobolev inequality  $f \in \mathcal{F}^{1,1}$ :

$$\int_{\mathbb{C}} |f(z)e^{-\frac{|z|^2}{2}}|^2 dA(z) - \pi|f(0)|^2 \leq (2\pi)^{-1} \left( \int_{\mathbb{C}} |f'(z)e^{-\frac{|z|^2}{2}}| dA(z) \right)^2$$

whose sharp form is

$$\int_{\mathbb{C}} |f(z)e^{-\frac{|z|^2}{2}}|^2 dA(z) - \pi|f(0)|^2 \leq (4\pi)^{-1} \left( \int_{\mathbb{C}} |f'(z)e^{-\frac{|z|^2}{2}}| dA(z) \right)^2$$

since this inequality can be proved valid for the entire functions  $f(z) = z^k$  with  $k = 1, 2, 3, \dots$  through a direct computation with the polar coordinate system, the mathematical induction and the inequality for the gamma function  $\Gamma(\cdot)$  below:

$$\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \leq \sqrt{\frac{k+1}{2}}.$$

The second is to decide: when  $\ln r \mapsto \ln M_{p,\alpha}(z^k, r)$  is convex for  $r \in (0, \infty)$ , namely, when the Gaussian Hadamard Three Circle Theorem below

$$\left( \ln \frac{r_2}{r_1} \right) \ln M_{p,\alpha}(z^k, r) \leq \left( \ln \frac{r_2}{r_1} \right) \ln M_{p,\alpha}(z^k, r_1) + \left( \ln \frac{r}{r_1} \right) \ln M_{p,\alpha}(z^k, r_2)$$

holds for  $0 < r_1 \leq r \leq r_2 < \infty$ . The expected result is presented in Theorem 7 of Section 3, saying that for a nonnegative integer  $k$  and a positive

number  $p$ ,

$$\begin{cases} \ln r \mapsto \ln M_{p,\alpha}(z^k, r) \text{ is concave as } r \in (0, \infty) \text{ under } 0 < \alpha < \infty; \\ \exists c \in (0, \infty) \ni \ln r \mapsto \ln M_{p,\alpha}(z^k, r) \text{ is convex and concave} \\ \text{as } r \in (0, c] \text{ and } r \in [c, \infty) \text{ respectively under } -\infty < \alpha \leq 0. \end{cases}$$

As a consequence, we have that if  $-\infty < \alpha, -p < 0$  then the function  $\ln r \mapsto \ln M_{p,\alpha}(z^k, r)$  is convex as  $r \in (0, \sqrt{(2+pk)/(-2\alpha)})]$  and hence the function  $\ln r \mapsto \ln M_{2,\alpha}(f, r)$  is convex as  $r \in (0, \sqrt{1/(-\alpha)})]$  for any entire function  $f : \mathbb{C} \mapsto \mathbb{C}$ . In other words,

$$\left( \ln \frac{r_2}{r_1} \right) \ln M_{2,\alpha}(f, r) \leq \left( \ln \frac{r_2}{r_1} \right) \ln M_{2,\alpha}(f, r_1) + \left( \ln \frac{r}{r_1} \right) \ln M_{2,\alpha}(f, r_2)$$

when  $0 < r_1 \leq r \leq r_2 < \sqrt{1/(-\alpha)}$ . However, as proved in Remark 9 via considering the entire function  $1+z$ , the last convexity cannot be extended to  $(0, \infty)$ .

## 2. TRACE INEQUALITIES FOR FOCK-SOBOLEV SPACES

We need two lemmas. The first lemma comes from [2] and [18, 17, 4, 11].

**Lemma 1.** *Let  $p, \sigma, a, t, \lambda \in (0, \infty)$ .*

(i) *If  $m$  is a nonnegative integer,  $p_m(z)$  is the Taylor polynomial of  $e^z$  of order  $m-1$  (with the convention that  $p_0 = 0$ ), and  $b > -(mp+2)$ , then*

$$\int_{\mathbb{C}} |e^{z\bar{w}} - p_m(z\bar{w})|^p e^{-a|w|^2} |w|^b dA(w) \lesssim |z|^b e^{\frac{p^2}{4a}|z|^2} \quad \forall \quad |z| \geq \sigma.$$

*Furthermore, this last inequality holds also for all  $z \in \mathbb{C}$  when  $b \leq pm$ .*

(ii) *If  $f : \mathbb{C} \mapsto \mathbb{C}$  is an entire function, then*

$$\left| f(z) e^{-\frac{\lambda}{2}|z|^2} \right|^p \lesssim \int_{B(z,t)} \left| f(w) e^{-\frac{\lambda}{2}|w|^2} \right|^p dA(w) \quad \forall \quad z \in \mathbb{C}.$$

(iii) *There exists a positive constant  $r_0$  such that for any  $0 < r < r_0$ , the Fock space  $\mathcal{F}^p$  exactly consists of all functions  $f = \sum_{w \in r\mathbb{Z}^2} c_w k_w$ , where*

$$\begin{cases} k_w(z) = \exp(z\bar{w} - |w|^2/2); \\ \{c_w : w \in r\mathbb{Z}^2\} \in l^p; \\ \|\{c_w\}\|_{l^p} = \left( \sum_{w \in r\mathbb{Z}^2} |c_w|^p \right)^{\frac{1}{p}}; \\ \mathbb{Z}^2 = \{n + im : n, m = 0, \pm 1, \pm 2, \dots\}; \\ r\mathbb{Z}^2 = \{r(n + im) : n, m = 0, \pm 1, \pm 2, \dots\}. \end{cases}$$

Moreover

$$\|f\|_{\mathcal{F}^p} \approx \inf \|\{c_w\}\|_{l^p} \quad \forall \quad f \in \mathcal{F}^p,$$

where the infimum is taken over all sequences  $\{c_w\}$  giving rise to the above decomposition.

The second lemma is the so-called Khinchine's inequality, which can be found, for example, in [7].

**Lemma 2.** *Suppose  $p \in (0, \infty)$  and  $c_j \in \mathbb{C}$ . For the integer part  $[t]$  of  $t \in (0, \infty)$  let*

$$r_0(t) = \begin{cases} 1, & 0 \leq t - [t] < 1/2 \\ -1, & 1/2 \leq t - [t] < 1 \end{cases}$$

and

$$r_j(t) = r_0(2^j t) \quad \forall \quad j = 1, 2, \dots$$

Then

$$\left( \sum_{j=1}^m |c_j|^2 \right)^{\frac{1}{2}} \approx \left( \int_0^1 \left| \sum_{j=1}^m c_j r_j(t) \right|^p dt \right)^{\frac{1}{p}}.$$

As the main result of this section, the forthcoming family of analytic-geometric trace inequalities for the Fock-Sobolev spaces is a natural generalization of the so-called diagonal Carleson measures for the Fock-Sobolev spaces in [2].

**Theorem 3.** *Let  $m$  be a nonnegative integer,  $r \in (0, \infty)$ , and  $\mu$  be a non-negative Borel measure on  $\mathbb{C}$ .*

(i) *If  $0 < p \leq q < \infty$ , then*

$$\|f\|_{L^q(\mathbb{C}, \mu)} \lesssim \|f\|_{\mathcal{F}^{p,m}} \quad \forall \quad f \in \mathcal{F}^{p,m}$$

*when and only when*

$$\sup_{a \in \mathbb{C}} \frac{\mu(B(a, r))}{(1 + |a|)^{mq}} < \infty.$$

*Equivalently,  $a \mapsto \mu(B(a, r))(1 + |a|)^{-qm}$  is of class  $L^\infty(\mathbb{C})$ .*

(ii) *If  $0 < q < p < \infty$ , then*

$$\|f\|_{L^q(\mathbb{C}, \mu)} \lesssim \|f\|_{p,m} \quad \forall \quad f \in \mathcal{F}^{p,m}$$

*when and only when*

$$\sum_{a \in s\mathbb{Z}^2} \left( \frac{\mu(B(a, r))}{(1 + |a|)^{mq}} \right)^{\frac{p}{p-q}} < \infty \quad \text{where } s \in (0, \infty).$$

*Equivalently,  $a \mapsto \mu(B(a, r))(1 + |a|)^{-qm}$  is of class  $L^{p/(p-q)}(\mathbb{C})$ .*

*Proof.* (i) Suppose  $0 < p \leq q < \infty$ . The following argument is similar to that of Theorem 10 in [2].

Assume firstly that  $\|f\|_{L^q(\mathbb{C}, \mu)} \lesssim \|f\|_{\mathcal{F}^{p,m}}$  holds for all  $f \in \mathcal{F}^{p,m}$ . Taking  $f = 1$  shows that  $\mu(K) \lesssim 1$  for any compact set  $K \subset \mathbb{C}$ .

Fix any  $a \in \mathbb{C}$  and let

$$f(z) = (e^{z\bar{a}} - p_m(z\bar{a}))/z^m$$

in the last assumption. Then Lemma 1 (i) implies

$$\int_{\mathbb{C}} \left| \frac{e^{z\bar{a}} - p_m(z\bar{a})}{z^m} e^{-\frac{1}{2}|z|^2} \right|^q d\mu(z) \lesssim (e^{\frac{p}{2}|a|^2})^{\frac{q}{p}} = e^{\frac{q}{2}|a|^2}.$$

In particular,

$$\int_{B(a,r)} \left| \frac{e^{z\bar{a}} - p_m(z\bar{a})}{z^m} e^{-\frac{1}{2}|z|^2} \right|^q d\mu(z) \lesssim e^{\frac{q}{2}|a|^2}.$$

If  $|a| > 2r$ , then  $|z|^m$  is comparable to  $(1 + |a|)^m$  for  $B(a, r)$ . So

$$\int_{B(a,r)} |e^{z\bar{a}}|^q |1 - e^{-z\bar{a}} p_m(z\bar{a})|^q e^{-\frac{q}{2}|z|^2} d\mu(z) \lesssim (1 + |a|)^{mq} e^{\frac{q}{2}|a|^2}$$

holds for all  $|a| > 2r$ . Note that

$$\lim_{|a| \rightarrow \infty} \inf_{z \in B(a,r)} |1 - e^{-z\bar{a}} p_m(z\bar{a})| = 1.$$

Thus

$$\int_{B(a,r)} |e^{z\bar{a}}|^q e^{-\frac{q}{2}|z|^2} d\mu(z) \lesssim (1 + |a|)^{mq} e^{\frac{q}{2}|a|^2}$$

holds for the sufficiently large  $|a|$ . But this last inequality is clearly true for smaller  $|a|$  as well. So we have

$$\int_{B(a,r)} |e^{z\bar{a}}|^q e^{-\frac{q}{2}|z|^2} d\mu(z) \lesssim (1 + |a|)^{mq} e^{\frac{q}{2}|a|^2} \quad \forall \quad a \in \mathbb{C}.$$

Completing a square in the exponent, we can rewrite the inequality above as

$$\int_{B(a,r)} e^{-\frac{q}{2}|z-a|^2} d\mu(z) \lesssim (1 + |a|)^{mq}$$

thereby deducing

$$\mu(B(a, r)) \lesssim (1 + |a|)^{mq} e^{\frac{q}{2}r^2} \quad \forall \quad a \in \mathbb{C}.$$

Conversely, assume that

$$\mu(B(a, r)) \lesssim (1 + |a|)^{mq} \quad \forall \quad a \in \mathbb{C}.$$

We proceed to estimate the integral

$$\|f\|_{L^q(\mathbb{C}, \mu)}^q = \int_{\mathbb{C}} |f(z) e^{-\frac{1}{2}|z|^2}|^q d\mu(z)$$

of any given function  $f \in \mathcal{F}^{p,m}$ . For any positive number  $s$  let  $Q_s$  denote the following square in  $\mathbb{C}$  with vertices  $0, s, si$ , and  $s + si$ :

$$Q_s = \{z = x + iy : 0 < x \leq s \text{ \& } 0 < y \leq s\}.$$

It is clear that

$$\mathbb{C} = \cup_{a \in s\mathbb{Z}^2} (Q_s + a)$$

is a decomposition of  $\mathbb{C}$  into disjoint squares of side length  $s$ . Thus

$$\|f\|_{L^q(\mathbb{C}, \mu)}^q = \sum_{a \in s\mathbb{Z}^2} \int_{Q_s+a} |f(z) e^{-\frac{1}{2}|z|^2}|^q d\mu(z).$$

Fix positive numbers  $s$  and  $t$  such that  $t + \sqrt{s} = r$ . By Lemma 1 (ii)

$$\begin{aligned} |f(z) e^{-\frac{1}{2}|z|^2}|^p &\lesssim \int_{B(z,t)} |f(w) e^{-\frac{1}{2}|w|^2}|^p dA(w) \\ &\lesssim \frac{1}{(1+|z|)^{mp}} \int_{B(z,t)} |w^m f(w) e^{-\frac{1}{2}|w|^2}|^p dA(w) \end{aligned}$$

holds for all  $z \in \mathbb{C}$ . Now if  $z \in Q_s + a$ , where  $a \in s\mathbb{Z}^2$  implies  $B(z, t) \subset B(a, r)$  by the triangle inequality, and hence  $1+|z| \approx 1+|a|$ . Consequently,

$$|f(z) e^{-\frac{1}{2}|z|^2}|^p \lesssim \frac{1}{(1+|a|)^{mp}} \int_{B(a,r)} |w^m f(w) e^{-\frac{1}{2}|w|^2}|^p dA(w).$$

This amounts to

$$|f(z) e^{-\frac{1}{2}|z|^2}|^q \lesssim \frac{1}{(1+|a|)^{mq}} \left( \int_{B(a,r)} |w^m f(w) e^{-\frac{1}{2}|w|^2}|^p dA(w) \right)^{\frac{q}{p}}.$$

Therefore,

$$\|f\|_{L^q(\mathbb{C}, \mu)}^q \lesssim \sum_{a \in s\mathbb{Z}^2} \frac{\mu(B(a, r))}{(1+|a|)^{mq}} \left( \int_{B(a,r)} |w^m f(w) e^{-\frac{1}{2}|w|^2}|^p dA(w) \right)^{\frac{q}{p}}.$$

Combining this last estimate with the previous assumption on  $\mu$  and  $p \leq q$ , we obtain

$$\begin{aligned} \|f\|_{L^q(\mathbb{C}, \mu)}^q &\lesssim \sum_{a \in s\mathbb{Z}^2} \left( \int_{B(a,r)} |w^m f(w) e^{-\frac{1}{2}|w|^2}|^p dA(w) \right)^{\frac{q}{p}} \\ &\lesssim \left( \sum_{a \in s\mathbb{Z}^2} \int_{B(a,r)} |w^m f(w) e^{-\frac{1}{2}|w|^2}|^p dA(w) \right)^{\frac{q}{p}}. \end{aligned}$$

Note that there exists a positive integer  $N$  such that each point in  $\mathbb{C}$  belongs to at most  $N$  of the disks  $B(a, r)$ , where  $a \in s\mathbb{Z}^2$ . So, one gets

$$\|f\|_{L^q(\mathbb{C}, \mu)}^q \lesssim \left( \int_{\mathbb{C}} |w^m f(w) e^{-\frac{1}{2}|w|^2}|^p dA(w) \right)^{\frac{q}{p}} \approx \|f\|_{\mathcal{F}_{p,m}}^q,$$

as desired.

(ii) Suppose  $0 < q < p < \infty$ . The following proof is inspired by [13].

First assume that  $\|f\|_{L^q(\mathbb{C}, \mu)} \lesssim \|f\|_{\mathcal{F}^{p,m}}$  holds for all  $f \in \mathcal{F}^{p,m}$ . For any  $\{c_j\} \in l^p$ , we may choose  $\{r_j(t)\}$  as in Lemma 2, thereby getting

$$\{c_j r_j(t)\} \in l^p \quad \& \quad \|\{c_j r_j(t)\}\|_{l^p} = \|\{c_j\}\|_{l^p}.$$

Then by Lemma 1 (iii) we know that

$$\sum_{j=1}^{\infty} c_j r_j(t) k_{a_j}(z) \equiv z^m f(z)$$

is in  $\mathcal{F}^p$  with norm  $\|f\|_{\mathcal{F}^{p,m}} \approx \inf \|\{c_j\}\|_{l^p}$ . Here  $\{a_j\}$  is the sequence of all complex numbers of  $s\mathbb{Z}^2$  and  $k_a(z) = e^{z\bar{a} - \frac{1}{2}|a|^2}$ . In particular,

$$f(z) = \sum_{j=1}^{\infty} c_j r_j(t) k_{a_j}(z) z^{-m}.$$

According to the assumption we have

$$\int_{\mathbb{C}} \left| \sum_{j=1}^{\infty} c_j r_j(t) \frac{k_{a_j}(z)}{z^m e^{|z|^2/2}} \right|^q d\mu(z) = \|f\|_{L^q(\mathbb{C}, \mu)}^q \lesssim \|f\|_{\mathcal{F}^{p,m}}^q,$$

whence getting by Lemma 2,

$$\int_{\mathbb{C}} \left( \sum_{j=1}^{\infty} |c_j k_{a_j}(z) e^{-\frac{1}{2}|z|^2}|^2 |z|^{-2m} \right)^{\frac{q}{2}} d\mu(z) \lesssim \|f\|_{\mathcal{F}^{p,m}}^q.$$

Also, note that if  $|a| > 2r$  then  $|z|^m$  is comparable to  $(1 + |a|)^m$  for  $z \in B(a, r)$ . So

$$\begin{aligned} & \int_{\mathbb{C}} \left( \sum_{j=1}^{\infty} |c_j k_{a_j}(z) e^{-\frac{1}{2}|z|^2}|^2 |z|^{-2m} \right)^{\frac{q}{2}} d\mu(z) \\ &= \sum_{l=1}^{\infty} \int_{Q_s + a_l} \left( \sum_{j=1}^{\infty} |c_j e^{-\frac{1}{2}|z - a_j|^2}|^2 |z|^{-2m} \right)^{\frac{q}{2}} d\mu(z) \\ &\geq \sum_{l=1}^{\infty} \int_{Q_s + a_l} |c_l|^q |z|^{-mq} e^{-\frac{q}{2}|z - a_l|^2} d\mu(z) \\ &\gtrsim \sum_{j=1}^{\infty} \int_{B(a_j, r)} |c_j|^q |z|^{-mq} e^{-\frac{q}{2}|z - a_j|^2} d\mu(z) \\ &\gtrsim \sum_{j=1}^{\infty} |c_j|^q \frac{\mu(B(a_j, r))}{(1 + |a_j|)^{mq}}. \end{aligned}$$



So, a combination of the previously-established inequalities gives

$$\sum_{j=1}^{\infty} |c_j|^q \frac{\mu(B(a_j, r))}{(1 + |a_j|)^{mq}} \lesssim \|\{|c_j|\}\|_{l^p}^q = \|\{|c_j|^q|\}\|_{l^{p/q}}.$$

Since  $p/(p - q)$  is the conjugate number of  $p/q$ , an application of the Riesz representation theorem yields

$$\left\{ \frac{\mu(B(a_j, r))}{(1 + |a_j|)^{mq}} \right\} \in l^{\frac{p}{p-q}}.$$

Conversely, assume that the last statement holds. Note that the first part of the argument for the above (i) tells that

$$\|f\|_{L^q(\mathbb{C}, \mu)}^q \lesssim \sum_{a \in s\mathbb{Z}^2} \frac{\mu(B(a, r))}{(1 + |a|)^{mq}} \left( \int_{B(a, r)} |w^m f(w) e^{-\frac{1}{2}|w|^2}|^p dA(w) \right)^{\frac{q}{p}}$$

holds for all  $f \in \mathcal{F}^{p,m}$ . Applying Hölder's inequality to the last summation we obtain

$$\begin{aligned} \|f\|_{L^q(\mathbb{C}, \mu)}^q &\lesssim \left( \sum_{a \in s\mathbb{Z}^2} \left( \frac{\mu(B(a, r))}{(1 + |a|)^{mq}} \right)^{\frac{p}{p-q}} \right)^{\frac{p-q}{p}} \\ &\quad \times \left( \sum_{a \in s\mathbb{Z}^2} \int_{B(a, r)} |w^m f(w) e^{-\frac{1}{2}|w|^2}|^p dA(w) \right)^{\frac{q}{p}}. \end{aligned}$$

Once again, notice that there exists a positive integer  $N$  such that each point in  $\mathbb{C}$  belongs to at most  $N$  of the disks  $B(a, r)$ , where  $a \in s\mathbb{Z}^2$ . So,

$$\|f\|_{L^q(\mathbb{C}, \mu)}^q \lesssim \left( \sum_{a \in s\mathbb{Z}^2} \left( \frac{\mu(B(a, r))}{(1 + |a|)^{mq}} \right)^{\frac{p}{p-q}} \right)^{\frac{p-q}{p}} \|f\|_{\mathcal{F}^{p,m}}^q.$$

This completes the argument.  $\square$

The following extends [1, Theorem 5] (cf. [10, Theorem 1]), and [3, Theorem 1], respectively.

**Corollary 4.** *Let  $\phi : \mathbb{C} \mapsto \mathbb{C}$  be an entire function. For  $p \in (0, \infty)$  and a nonnegative integer  $m$  define two linear operators acting on an entire function  $f : \mathbb{C} \mapsto \mathbb{C}$ :*

$$\begin{cases} C_\phi f(z) = f \circ \phi(z) & \forall \quad z \in \mathbb{C}; \\ T_\phi f(z) = \int_0^z f(w) \phi'(w) dw & \forall \quad z \in \mathbb{C}. \end{cases}$$

(i) *The composition operator  $C_\phi : \mathcal{F}^{p,m} \mapsto \mathcal{F}^q$  exists as a bounded operator if and only if*

$$\begin{cases} \sup_{a \in \mathbb{C}} \frac{\int_{\phi^{-1}(B(a,r))} e^{-q|z|^2/2} dA(z)}{(1+|a|)^{mq}} < \infty \text{ when } 0 < p \leq q < \infty; \\ \int_{\mathbb{C}} \left( \frac{\int_{\phi^{-1}(B(a,r))} e^{-q|z|^2/2} dA(z)}{(1+|a|)^{mq}} \right)^{p/(p-q)} dA(a) < \infty \text{ when } 0 < q < p < \infty. \end{cases}$$

(ii) *The Riemann-Stieltjes integral operator  $T_\phi : \mathcal{F}^{p,m} \mapsto \mathcal{F}^q$  exists as a bounded operator if and only if*

$$\begin{cases} \sup_{a \in \mathbb{C}} \frac{\int_{B(a,r)} \left( \frac{|\phi'(z)|}{(1+|z|)} \right)^q dA(z)}{(1+|a|)^{mq}} < \infty \text{ when } 0 < p \leq q < \infty; \\ \int_{\mathbb{C}} \left( \frac{\int_{B(a,r)} \left( \frac{|\phi'(z)|}{(1+|z|)} \right)^q dA(z)}{(1+|a|)^{mq}} \right)^{p/(p-q)} dA(a) < \infty \text{ when } 0 < q < p < \infty. \end{cases}$$

*Proof.* (i) For any Borel set  $E \subset \mathbb{C}$  let  $\phi^{-1}(E)$  be the pre-image of  $E$  under  $\phi$  and

$$\mu(E) = \int_{\phi^{-1}(E)} \exp\left(-\frac{q|z|^2}{2}\right) dA(z).$$

Then

$$\|C_\phi f\|_{L^q(\mathbb{C}, \mu)}^q = \int_{\mathbb{C}} |f(z) e^{-\frac{|z|^2}{2}}|^q d\mu(z) \quad \forall \quad f \in \mathcal{F}^{p,m}.$$

An application of Theorem 3 with the above formula gives the desired result.

(ii) According to [3, Proposition 1], an entire function  $f : \mathbb{C} \mapsto \mathbb{C}$  belongs to  $\mathcal{F}^q$  if and only if

$$\int_{\mathbb{C}} \left( \frac{|f'(z)| e^{-\frac{|z|^2}{2}}}{1+|z|} \right)^q dA(z) < \infty.$$

So,  $T_\phi f \in \mathcal{F}^q$  is equivalent to

$$\int_{\mathbb{C}} \left( \frac{|f(z)\phi'(z)| e^{-\frac{|z|^2}{2}}}{1+|z|} \right)^q dA(z) < \infty.$$

Now, choosing

$$d\mu(z) = \left( \frac{|\phi'(z)|}{1+|z|} \right)^q dA(z)$$

in Theorem 3, we get the boundedness result for  $T_\phi$ . □

## 3. CONVEXITIES OR CONCAVITIES IN LOGARITHM

We also need two lemmas. The first one comes directly from [12, Lemmas 2, 1, 6] with  $(0, 1)$  being replaced by  $(0, \infty)$ .

**Lemma 5.**

(i) Suppose  $f$  is positive and twice differentiable on  $(0, \infty)$ . Let

$$D(f(x)) \equiv \frac{f'(x)}{f(x)} + x \frac{f''(x)}{f(x)} - x \left( \frac{f'(x)}{f(x)} \right)^2.$$

Then the function  $\ln f(x)$  is concave in  $\ln x$  if and only if  $D(f(x)) \leq 0$  on  $(0, \infty)$  and  $\ln f(x)$  is convex in  $\ln x$  if and only if  $D(f(x)) \geq 0$  on  $(0, \infty)$ .

(ii) Suppose  $f$  is twice differentiable on  $(0, \infty)$ . Then  $f(x)$  is convex in  $\ln x$  if and only if  $f(x^2)$  is convex in  $\ln x$  and  $\ln f(x)$  is concave in  $\ln x$  if and only if  $f(x^2)$  is concave in  $\ln x$ .

(iii) Suppose  $\{h_k(x)\}$  is a sequence of positive and twice differentiable functions on  $(0, \infty)$  such that the function

$$H(x) = \sum_{k=0}^{\infty} h_k(x)$$

is also twice differentiable on  $(0, \infty)$ . If for each natural number  $k$  the function  $\ln h_k(x)$  is convex in  $\ln x$ , then  $\ln H(x)$  is also convex in  $\ln x$ .

The second lemma as below is elementary.

**Lemma 6.** Suppose  $f$  is continuous differentiable on  $[0, \infty)$ . If  $f'(\infty) \equiv \lim_{x \rightarrow \infty} f'(x) = -\infty$ , then  $f(\infty) \equiv \lim_{x \rightarrow \infty} f(x) = -\infty$ .

The main result of this section is the following log-convexity theorem.

**Theorem 7.** Suppose  $k$  is a nonnegative integer and  $0 < p < \infty$ .

- (i) If  $0 < \alpha < \infty$ , then the function  $r \mapsto \ln M_{p,\alpha}(z^k, r)$  is concave in  $\ln r$ .
- (ii) If  $-\infty < \alpha \leq 0$ , then there exists some  $c$  (depending on  $k$  and  $\alpha$ ) on  $(0, \infty)$  such that the function  $r \mapsto \ln M_{p,\alpha}(z^k, r)$  is convex in  $\ln r$  on  $(0, c]$  and concave in  $\ln r$  on  $[c, \infty)$ .

*Proof.* The case  $\alpha = 0$  is a straightforward by-product of the classical Hardy convexity theorem (cf. [9]). So, for the rest of the proof we may assume  $\alpha \neq 0$ .

By the polar coordinates and an obvious change of variables, we have

$$M_{p,\alpha}(z^k, r) = \frac{\int_0^{r^2} t^{pk/2} e^{-\alpha t} dt}{\int_0^{r^2} e^{-\alpha t} dt}.$$

For any nonnegative parameter  $\lambda$  we define

$$f_\lambda(x) = \int_0^x t^\lambda e^{-\alpha t} dt \quad \forall x \in (0, \infty).$$

To prove Theorem 7, by Lemma 5 (i)-(ii), we need only to consider the function

$$\Delta(\lambda, x) = \frac{f'_\lambda}{f_\lambda} + x \frac{f''_\lambda}{f_\lambda} - x \left( \frac{f'_\lambda}{f_\lambda} \right)^2 - \left( \frac{f'_0}{f_0} + x \frac{f''_0}{f_0} - x \left( \frac{f'_0}{f_0} \right)^2 \right).$$

Here and henceforth, the derivatives  $f'_\lambda(x)$  and  $f''_\lambda(x)$  are taken with respect to  $x$  not  $\lambda$ .

To simplify notation, we write  $h = f_\lambda(x)$  and denote by  $h'$ ,  $h''$ ,  $h'''$  to the various derivatives of  $f_\lambda(x)$  with respect to  $x$ . Meanwhile,  $\partial/\partial\lambda$  stands for the derivative with respect to  $\lambda$ .

Thanks to  $h = \int_0^x t^\lambda e^{-\alpha t} dt$ , we get

$$\begin{cases} h' = x^\lambda e^{-\alpha x}; \\ h'' = (\lambda - \alpha x) x^{\lambda-1} e^{-\alpha x}; \\ h''' = x^{\lambda-2} e^{-\alpha x} (\lambda^2 - \lambda - 2\lambda\alpha x + \alpha^2 x^2). \end{cases}$$

At the same time, we have

$$\begin{cases} \frac{\partial h}{\partial \lambda} = \int_0^x t^\lambda e^{-\alpha t} \ln t dt; \\ \frac{\partial h'}{\partial \lambda} = \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial \lambda} \right) = h' \ln x; \\ \frac{\partial h''}{\partial \lambda} = \frac{h'}{x} + h'' \ln x. \end{cases}$$

Note that the function inside the brackets in  $\Delta(\lambda, x)$  is independent of  $\lambda$ . So,

$$\begin{aligned} & \frac{\partial \Delta}{\partial \lambda} \\ &= \frac{1}{h^2} \left( h \frac{\partial h'}{\partial \lambda} + x h \frac{\partial h''}{\partial \lambda} - 2x h' \frac{\partial h'}{\partial \lambda} \right) \\ & \quad - \frac{1}{h^3} \frac{\partial h}{\partial \lambda} (h h' + x h h'' - 2x (h')^2) \\ &= \frac{1}{h^2} (h h' \ln x + h h' + x h h'' \ln x - 2x (h')^2 \ln x) \\ & \quad - \frac{1}{h^3} \frac{\partial h}{\partial \lambda} (h h' + x h h'' - 2x (h')^2) \\ &= \frac{h'}{h} + \frac{1}{h^3} \left( h \ln x - \frac{\partial h}{\partial \lambda} \right) (h h' + x h h'' - 2x (h')^2). \end{aligned}$$

From now on, we use the notation  $X \sim Y$  to represent that  $X$  and  $Y$  have the same sign. Let us consider the following two functions (with  $\lambda$  fixed):

$$\begin{cases} d_1(x) = h \ln x - \frac{\partial h}{\partial \lambda}; \\ d_2(x) = \frac{hh' + xhh'' - 2x(h')^2}{h'} = (\lambda + 1 - \alpha x)h - 2x^{\lambda+1}e^{-\alpha x}. \end{cases}$$

Since  $d_1'(x) = h/x > 0$ , one has  $d_1(x) \geq d_1(0) = 0$ . Now we want to prove that  $d_2(x) < 0$  for all  $x > 0$ . By direct computations, we obtain

$$\begin{cases} d_2'(x) = -\alpha h - (\lambda + 1 - \alpha x)x^\lambda e^{-\alpha x}; \\ d_2''(x) = (\lambda + 1 - \alpha x)(-\lambda + \alpha x)x^{\lambda-1}e^{-\alpha x}. \end{cases}$$

(i) If  $\alpha > 0$ , then under  $0 < x \leq \frac{\lambda+1}{\alpha}$  we have  $d_2'(x) \leq -\alpha h < 0$ . When  $x > \frac{\lambda+1}{\alpha}$ , it is easy to obtain  $d_2''(x) < 0$ , and then

$$d_2'(x) \leq d_2'\left(\frac{\lambda+1}{\alpha}\right) = -\alpha h \left(\frac{\lambda+1}{\alpha}\right) < 0.$$

Hence  $d_2(x) < d_2(0) = 0$  for all  $x > 0$ .

(ii) If  $\alpha < 0$ , then it is easy to see  $d_2''(x) < 0$ , and hence  $d_2'(x) \leq d_2'(0) = 0$ . This in turn implies  $d_2(x) < d_2(0) = 0$  for all  $x > 0$ .

With the help of the above analysis, we deduce

$$\frac{\partial \Delta}{\partial \lambda} \sim -\frac{h^2 h'}{hh' + xhh'' - 2x(h')^2} - h \ln x + \frac{\partial h}{\partial \lambda} =: \delta(x).$$

Further computations derive

$$\begin{aligned} \delta'(x) &= -\frac{2h(h')^2 + h^2 h''}{hh' + xhh'' - 2x(h')^2} \\ &\quad + \frac{h^2 h'(2hh'' + xhh''' - 3xh'h'' - (h')^2)}{(hh' + xhh'' - 2x(h')^2)^2} - \frac{h}{x} \\ &= \left( \frac{h^2}{x(hh' + xhh'' - 2x(h')^2)^2} \right) \\ &\quad \times \left( -((h')^2 + xh'h'' + 2x^2(h'')^2 - x^2h'h''')h + x(h')^2(h' + xh'') \right) \\ &= \left( \frac{hh'}{hh' + xhh'' - 2x(h')^2} \right)^2 \\ &\quad \times \left( \frac{((\lambda+1)^2 - (2\lambda+1)\alpha x + \alpha^2 x^2) \delta_1(x)}{x} \right). \end{aligned}$$

Here

$$\delta_1(x) = -h + \frac{x^{\lambda+1}e^{-\alpha x}(\lambda+1-\alpha x)}{(\lambda+1)^2 - (2\lambda+1)\alpha x + \alpha^2 x^2}.$$

And, a computation implies

$$\delta'_1(x) = \frac{-\alpha x^{\lambda+1} e^{-\alpha x} (\lambda + 1 + \alpha x)}{((\lambda + 1)^2 - (2\lambda + 1)\alpha x + \alpha^2 x^2)^2}$$

and then

$$\delta'_1(0) = \delta_1(0) = 0.$$

With details deferred to after the proof, we also have  $\delta'(0) = 0$  and when  $\alpha < 0$  we have  $\delta'(\infty) = -\infty$ . Without loss of generality, we may just handle the case  $\lambda > 0$  in what follows.

(i) If  $\alpha > 0$ , then  $\delta'_1(x) < 0$  for all  $x \in (0, \infty)$ , and hence  $\delta_1(x) < \delta_1(0) = 0$  on  $(0, \infty)$ . This implies

$$\frac{\partial \Delta(\lambda, x)}{\partial \lambda} < 0 \quad \forall \quad x \in (0, \infty).$$

Therefore,  $\Delta(\lambda, x) \leq \Delta(0, x) = 0$ , and the desired result follows.

(ii) If  $\alpha < 0$ , then  $\delta'_1(x)$  has only one zero  $-(\lambda + 1)/\alpha$  on  $(0, \infty)$  and  $\delta_1(x)$  is increasing on  $(0, -(\lambda + 1)/\alpha)$  and decreasing on  $(-(\lambda + 1)/\alpha, \infty)$ . Noticing  $\delta'_1(\infty) = -\infty$ , we use Lemma 6 to get  $\delta_1(\infty) = -\infty$ . Hence  $\delta_1(x)$  has only one zero  $x^*$  on  $(0, \infty)$  (Note that  $x^* > \frac{\lambda+1}{-\alpha}$ .) and  $\delta_1(x)$  is positive on  $(0, x^*)$  and negative on  $(x^*, \infty)$ . For  $\delta(x)$  and  $\frac{\partial \Delta}{\partial \lambda}$  we have similar results. Hence  $\frac{\partial \Delta}{\partial \lambda}$  has exactly one zero  $x_0$  (depending on  $k$  and  $\alpha$ ) on  $(0, \infty)$  (Note that  $x_0 > x^* > \frac{\lambda+1}{-\alpha}$ .) and  $\frac{\partial \Delta}{\partial \lambda}$  is positive on  $(0, x_0)$  and negative on  $(x_0, \infty)$ . This implies  $\Delta(\lambda, x) \geq \Delta(0, x) = 0$  on  $(0, x_0)$  and  $\Delta(\lambda, x) \leq \Delta(0, x) = 0$  on  $(x_0, \infty)$ . Now, setting  $c = \sqrt{x_0}$  yields the desired result.

Finally, let us verify the above-claimed formulas:

$$\begin{cases} \delta'(0) = 0; \\ \delta'(\infty) = -\infty. \end{cases}$$

As a matter of fact, L'Hopital's rule gives

$$\lim_{x \rightarrow 0} \frac{h}{x} = 0, \quad \lim_{x \rightarrow 0} \frac{xh'}{h} = \lim_{x \rightarrow 0} \frac{h' + xh''}{h'} = \lambda + 1.$$

Consequently,

$$\lim_{x \rightarrow 0} \frac{hh'}{hh' + xhh'' - 2x(h')^2} = \lim_{x \rightarrow 0} \frac{1}{\frac{h' + xh''}{h'} - 2\frac{xh'}{h}} = -(\lambda + 1)^{-1}.$$

It follows from the definition of  $\delta_1(x)$  that  $\lim_{x \rightarrow 0} \frac{\delta_1(x)}{x} = 0$ . So, by the definition of  $\delta_1(x)$  we have  $\delta'(0) = 0$ .

In a similar manner, another application of L'Hopital's rule derives

$$\lim_{x \rightarrow \infty} \frac{h'}{h} = \lim_{x \rightarrow \infty} \frac{h''}{h'} = -\alpha$$

and consequently,

$$\lim_{x \rightarrow \infty} \frac{xhh'}{hh' + xhh'' - 2x(h')^2} = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x} + \frac{h''}{h'} - 2\frac{h'}{h}} = \frac{1}{\alpha}.$$

The definition of  $\delta_1(x)$  and L'Hopital's rule imply

$$\lim_{x \rightarrow \infty} \frac{\delta_1(x)}{x} = \lim_{x \rightarrow \infty} \delta_1'(x) = -\infty,$$

and then  $\delta'(\infty) = -\infty$ .  $\square$

**Corollary 8.** *Suppose  $\alpha < 0$  and  $0 < p < \infty$ . Then the function  $r \mapsto \ln M_{p,\alpha}(z^k, r)$  is convex in  $\ln r$  on  $(0, c]$ , where  $c = \sqrt{(pk+2)(-2\alpha)^{-1}}$ . Moreover, for  $p = 2$ , the function  $r \mapsto \ln M_{2,\alpha}(f, r)$  is convex in  $\ln r$  on  $(0, \sqrt{(-\alpha)^{-1}}]$  for any entire function  $f : \mathbb{C} \mapsto \mathbb{C}$ .*

*Proof.* The first part of Corollary 8 follows from the proof of Theorem 7 with  $\lambda = pk/2$ . As for the second part, it is easy to see  $c \geq \sqrt{\frac{1}{-\alpha}}$ . Suppose

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

It follows from an integration in polar coordinates that

$$M_{2,\alpha}(f, r) = \sum_{k=0}^{\infty} |a_k|^2 M_{2,\alpha}(z^k, r).$$

Now, applying Lemma 5 (iii) we obtain the desired result.  $\square$

**Remark 9.** *Theorem 7 tells us that the integral means of all monomials are logarithmically concave when  $\alpha > 0$ . However, this is not true for all entire functions, even for linear mappings.*

*Proof.* For instance, just choose  $p = 2, \alpha = 1$  and  $f(z) = a + z$ . Using polar coordinates and changing variables we have

$$M_{p,\alpha}(f, r) = \frac{\int_0^{r^2} (c+t)e^{-t} dt}{\int_0^{r^2} e^{-t} dt} \quad \text{where } c = |a|^2.$$

By Lemma 5(i), we just need to consider the function

$$F(x) = \frac{\int_0^x (c+t)e^{-t} dt}{\int_0^x e^{-t} dt} = \frac{c+1 - (c+1+x)e^{-x}}{1-e^{-x}} \equiv \frac{g(x)}{h(x)}.$$

Employing the  $D$ -notation in Lemma 5 (i), we have

$$\begin{cases} D(g(x)) = \frac{1}{g^2} ((c+2x-cx-x^2)e^{-x}g(x) - x(c+x)^2e^{-2x}) \\ \text{and} \\ D(h(x)) = \frac{e^{-x}}{h^2}(1-x-e^{-x}), \end{cases}$$

whence getting

$$\begin{aligned}
& D(F(x)) \\
&= D(g(x)) - D(h(x)) \\
&= e^{-x} ((c+1)(-1+3x-x^2) + (3+3c-6x-6cx-x^2)e^{-x} \\
&\quad + (-3-3c+3x+3cx+2x^2+cx^2+x^3)e^{-2x} + (c+1)e^{-3x}) \\
&\sim (c+1)(-1+3x-x^2)e^{3x} + (3+3c-6x-6cx-x^2)e^{2x} \\
&\quad + (-3-3c+3x+3cx+2x^2+cx^2+x^3)e^x + (c+1) \\
&\equiv G(x).
\end{aligned}$$

A direct computation gives

$$\begin{cases} G'(x) \\ \sim (c+1)(7-3x) - \frac{14+12c+2x}{e^x} + \frac{7+5c+5x+cx+x^2}{e^{2x}} \equiv H(x); \\ H'(x) = -3(c+1) + \frac{12+12c+2x}{e^x} - \frac{9+9c+8x+2cx+2x^2}{e^{2x}}; \\ H''(x) \sim -10-12c-2x + \frac{10+16c+12x+4cx+4x^2}{e^x} \equiv J(x); \\ J'(x) = -2(1-e^{-x}) - (12c+4x+4cx+4x^2)e^{-x} \leq 0. \end{cases}$$

Noticing that  $J(0) = 4c > 0$  and  $J(\infty) = -\infty$ , we know that there exists a number  $x_3 \in (0, \infty)$  such that  $J(x)$  is positive, and hence  $H''(x)$  is positive on  $(0, x_3)$  and negative on  $(x_3, \infty)$ . Note that

$$\begin{cases} H'(0) = H(0) = G(0) = 0; \\ H'(\infty) = -3(c+1) < 0; \\ H(\infty) = G(\infty) = -\infty. \end{cases}$$

So, the functions  $H''$ ,  $H'$ ,  $H$ ,  $G$  have similar properties. In particular, there exists a  $x_0 \in (0, \infty)$  such that  $G(x)$ , and hence,  $D(F(x))$  is positive on  $(0, x_0)$  and negative on  $(x_0, \infty)$ . This implies that  $\ln M_{p,\alpha}(f, r)$  is convex in  $\ln r$  on  $(0, \sqrt{x_0})$  and concave in  $\ln r$  on  $(\sqrt{x_0}, \infty)$ . Especially, when  $c = 0$ , the function  $\ln M_{p,\alpha}(f, r)$  is concave in  $\ln r$  on  $(0, \infty)$ .

Certainly, it is interesting to determine the maximal interval  $(\lambda, \infty)$  on which  $G$  is negative, that is, the area integral means  $M_{p,\alpha}(a+z, r)$  is logarithmically concave on  $(\sqrt{\lambda}, \infty)$  for any  $a \in \mathbb{C}$ .

It follows from the definition of  $G$  that

$$G(x) \sim G_0(x) + \frac{x^2 e^x}{c+1} (1+x-e^x),$$

where

$$G_0(x) = (-1+3x-x^2)e^{3x} + (3-6x)e^{2x} + (-3+3x+x^2)e^x + 1.$$

Note that

$$G(x) < 0 \quad \forall \quad c = |a|^2 \iff G_0(x) < 0.$$



Also, it is not hard to prove that  $G_0(x)$  has exactly one real zero  $\lambda$  in  $(0, \infty)$ , and  $G_0(x)$  is positive on  $(0, \lambda)$  and negative on  $(\lambda, \infty)$ . A numerical computation shows that  $\lambda = 1.86047095 \dots$ . This implies that  $M_{2,1}(a + z, r)$  is logarithmically concave on  $(\sqrt{\lambda}, \infty)$  for any  $a \in \mathbb{C}$ , and the interval  $(\sqrt{\lambda}, \infty)$  is maximal.  $\square$

## REFERENCES

- [1] B. J. Carswell, B. D. MacCluer and A. Schuster, Composition operators on the Fock space, *Acta Sci. Math. (Szeged)* 69 (2003), 871-887.
- [2] H. R. Cho and K. Zhu, Fock-Sobolev spaces and their Carleson measures, *J. Funct. Anal.* 263 (2012), 2483-2506.
- [3] O. Constantin, A Volterra-type integration operator on Fock spaces, *Proc. Amer. Math. Soc.* 140 (2012), 4247-4257.
- [4] S. Janson, J. Peetre and R. Rochberg, Hankel forms and the Fock space, *Revista Mat. Ibero-Amer.* 3 (1987), 61-138.
- [5] M. Ledoux, *Isoperimetry and Gaussian Analysis*, École d'Été de Probabilités de Saint-Flour 1994.
- [6] D. H. Luecking, Multipliers of Bergman spaces into Lebesgue spaces, *Proc. Edinburgh Math. Soc.* 29 (1986), 125-131.
- [7] D. H. Luecking, Embedding theorems for spaces of analytic functions via Khinchine's inequality, *Michigan Math. J.* 40 (1993), 333-358.
- [8] J. M. Pearson, The Poincaré inequality and entire functions, *Proc. Amer. Math. Soc.* 118 (1993), 1193-1197.
- [9] A. E. Taylor, New proofs of some theorems of Hardy by Banach space methods, *Math. Maga.* 23 (1950), 115-124.
- [10] S.-I. Ueki, Weighted composition operator on the Fock space, *Proc. Amer. Math. Soc.* 135 (2007), 1405-1410.
- [11] R. Wallstén, The  $S^p$ -criterion for Hankel forms on the Fock space,  $0 < p < 1$ , *Math. Scand.* 64 (1989), 123-132.
- [12] C. Wang and K. Zhu, Logarithmic convexity of area integral means for analytic functions, *Math. Scand.* (in press).
- [13] J. Xiao, Riemann-Stieltjes operators between weighted Bergman spaces, in *Complex and Harmonic Analysis*, pp. 205-212, DEStech, Lancaster, UK, 2007.
- [14] J. Xiao and W. Xu, Weighted integral means of mixed areas and lengths under holomorphic mappings, *arXiv: 1105.6042v1[math.CV]*30May2011.
- [15] J. Xiao and K. Zhu, Volume integral means of holomorphic functions, *Proc. Amer. Math. Soc.* 139 (2011), 1455-1465.
- [16] S.-T. Yau, Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold, *Ann. scient. Éc. Norm. Sup.* 8 (1975), 478-507.
- [17] K. Zhu, Translation invariance of Fock spaces, *arXiv:1101.4201v1[math.CV]*21Jan2011.
- [18] K. Zhu, *Analysis on Fock spaces*, Springer-Verlag, New York, 2012.

CHUNJIE WANG, DEPARTMENT OF MATHEMATICS, HEBEI UNIVERSITY OF TECHNOLOGY, TIANJIN 300401, CHINA

*E-mail address:* wcj@hebut.edu.cn

JIE XIAO, DEPARTMENT OF MATHEMATICS AND STATISTICS, MEMORIAL UNIVERSITY OF NEWFOUNDLAND, ST. JOHN'S, NL A1C 5S7, CANADA

*E-mail address:* jxiao@mun.ca